LETTERS

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Tumbling cards

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When a stiff rectangular card is dropped in still air with its long axis horizontal, it often settles into a regular mode of motion; while revolving around its long axis it descends along a path that is inclined to the vertical at a nearly constant angle. We show experimentally that the tumbling frequency Ω of a card of length l, width w and thickness d ($l \gg w \gg d$) scales as $\Omega \sim d^{1/2} w^{-1}$, consistent with a simple dimensional argument that balances the drag against gravity. © *1999 American Institute of Physics*. [S1070-6631(99)02401-0]

A flat, thin rectangular card released in still air with its long axis horizontal will either flutter to the ground, periodically oscillating from side to side, or tumble while drifting steadily to one side. Since the first studies by Clerk-Maxwell,[1,2] the phenomenon and its variants, collectively known by the term *autorotation* have been investigated in a variety of contexts, including soaring flight,[3] unsteady aerodynamics,[4–6] reentry-vehicle dynamics,[7] power generation by wind,[8] and seed dispersal.[9] The type of motion is not restricted to tumbling or oscillations; for example, a card with a shape of an isoceles triangle falls along a helical path while simultaneously tumbling around its axis of symmetry. In addition to the symmetries of the falling object which play an important role in determining the behavior of a falling object,[10] the physical parameters in the problem are the object's dimensions, the densities of the fluid and solid, gravity, and the viscosity of the fluid.

For the falling rectangular strip, these parameters are its length l, width w, and thickness d ($l > w \gg d$), the densities of the material of the card ρ_s and of air $\rho_f = 1.23 \times 10^{-3}$ g/cm^3, gravity g, and the viscosity of air $\mu = 1.8 \times 10^{-4}$ g/cm s. Then a dimensionless characterization of these parameters may be achieved in terms of the aspect ratios, $w/d, l/w$, the buoyancy number $\rho_s/\rho_f - 1$, and a Reynolds number $\rho_f \Omega w^2/\mu$. For sufficiently large $w/d, l/w$, a falling strip will always settle into a steady tumbling and drifting motion.[1] In Fig. 1, we show this mode using a sequence of superposed video images; the strip tumbles while drifting to the right, while the axis of rotation always points out of the plane of the paper. As the card starts from a nearly vertical position it slices through the air, rapidly moving downwards. Following the observations of Maxwell[1] we see that this leads to two high pressure regions, one below the card close to the leading edge, and one above the card close to the trailing edge. The resulting torque rotates the card counterclockwise, as it begins to move broadside-on and slows down. If the inertial forces are large enough, the card continues past the broadside-on position while speeding up until it is nearly vertical again. Then it starts to slice through the air vertically and the scenario is repeated. Because of the unsteady nature of this rotation, which is fast as the card goes from vertical to horizontal and slow in the opposite case, there is also a net horizontal force on the card that causes the tumbling card to drift to the right.

There has been a resurgence of interest in this problem, beginning with the qualitative theory of Ref. 11 (but see Ref. 12). In Ref. 13, a different qualitative theory was proposed to account for the anisotropic added mass of a thin card. Since then, experiments[14] have mapped out the phase diagram for falling disks showing the regions where one observes the different modes of motion. More recent experiments[15] have focused on the transition between the tumbling and oscillations of a falling strip confined to a vertical Hele–Shaw cell. The results in Ref. 15 show that the aspect ratio w/d of the card determines whether it will tumble or oscillate (flutter), in agreement with Ref. 13 (the Froude number used in Ref. 15 is equivalent to the aspect ratio used in Ref. 13 scaled by the buoyancy number). Thus strips with a rectangular cross section of large enough aspect ratio tumble, while strips with a cross section that is nearly circular oscillate. A modified version of the theory in Ref. 11 was used in Ref. 15, and also captures the transition qualitatively. That all these models capture the transition is perhaps not surprising, since they all have the three necessary dynamical ingredients for this type

FIG. 1. Consecutive images from a video of a thin card tumbling counterclockwise through air. The card dimensions are approximately 75 mm $\times$15 mm$\times$0.1 mm, the density of the card material is 1.34 g/cm^3, the tumbling frequency is approximately 15 Hz, and the individual images are 1/60 s apart. The apparent changes in the dimensions of the card are illusions of perspective.

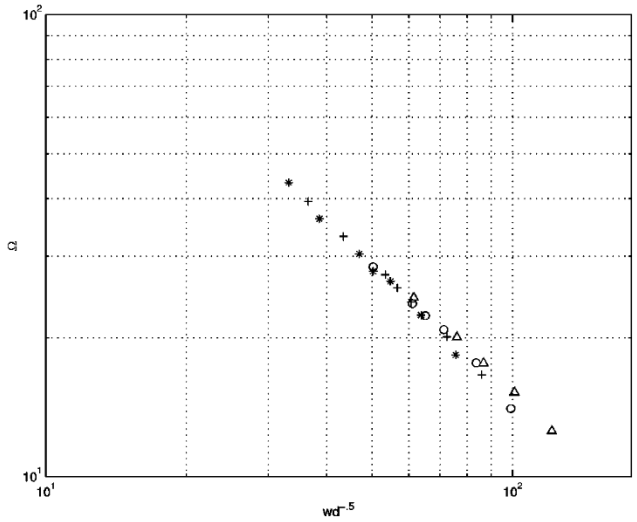

FIG. 2. Log–log plot of the tumbling frequency Ω of 75 *mm* long cards vs the normalized width $wd^{-1/2}$. The experiments were carried out for strips of four different thicknesses $d=0.051$ mm ($\triangle$), $d=0.076$ mm ($\bigcirc$), $d=0.102$ mm, (+), $d=0.127$ mm (*), and w is varied from 11 to 27 mm in steps of 3 mm for each thickness. The data are fit by the power law $\Omega \sim d^{0.5\pm0.03} w^{-1\pm0.03}$.

of behavior: (a) two equilibria corresponding to the card falling in its (stable) broadside-on and (unstable) end-on configurations, (b) a solution that connects these equilibria guaranteed by the periodicity of the angle characterizing the orientation of the card, and (c) a means of bringing the two equilibria together as some parameter such as w/d or the drag coefficient is changed. This last ingredient is where the individual models differ; however they all lead to a saddle-node bifurcation[16] beyond which only the tumbling mode is stable.[17] Since studies focusing on the flutter–tumble transition alone do not characterize the physics completely, we have experimentally probed the tumbling of strips far from this transition.

Our experiments were performed using long rectangular strips cut from reflective plastic shimstock. We were limited in our choice of strips by two factors: If the strips were too thin, they began to bend so that new effects were introduced into the problem, while if they were too thick, they did not reach a steady tumbling rate when dropped from a height of 1 m. The strips were electromagnetically clamped to the edge of a platform and released so that the tipping moment due to the weight was sufficient to make them tumble immediately. A halogen lamp was used to illuminate them and the intensity fluctuations of the reflected light were measured by a photodetector. Both the lamp and photodetector were placed directly in the path of descent of the strip and pointed toward it so that the orientation of the strip that delivered the light to the photodetector did not change through the fall. Only data in which the tumbling strips had reached steady state were analyzed. We only measured the average tumbling frequency; there is a second frequency associated with the speeding up and slowing down of the card which is twice the tumbling frequency. In order to minimize the effects of air currents we used heat-filtered light and confined the experiments to a large box of height 1 m. The width w of the cards was varied from 11 to 27 mm in steps of approximately 3 mm and cards of four different thickness 0.051, 0.076, 0.102 and 0.127 mm were used. Varying the length of the card did not change the frequency of rotation indicating that the cards were sufficiently long (75 mm) to eliminate three-dimensional effects. In Fig. 2 we plot the tumbling frequency Ω as a function of the normalized width $W \equiv wd^{-1/2}$ and find that $\Omega \sim W^{-1\pm0.03}$ is a good fit to the data. This scaling law is valid in a regime far from the onset of tumbling and also far from the regime when the strip becomes extremely flexible and starts to bend;[4] this is probably why the dimensionless constant in the scaling law is $O(10^2)$. The angle of the path of descent varies very slowly in this parameter regime[4] and is approximately $36° \pm 3°$. It is determined by the relative ratio of the lift and drag forces on the strip; as it starts to spin rapidly, a cylindrical volume of fluid is kept in motion so that the effective lift and drag are approximately equal. This would lead to an angle of descent that is close to 45°; deviations from this are due to the dynamics of vortex shedding at the trailing edge[4,15] of the strip.

In our experiments, $\Omega \sim 15$ Hz, $w \sim 15$ mm so that Re $\sim 10^2$; then fluid forces may be most simply accounted for in terms of a Reynolds-number-independent drag law.[18] In this high Reynolds number regime the drag force is proportional to the square of the velocity, with nearly constant drag coefficients. However, because of the shape anisotropy of the card the drag force depends on the direction of movement. Similarly, the inertial forces on the card also depend on the direction of motion. This can be quantified in terms of a shape-dependent anisotropic effective-mass tensor[19,13] that has two contributions; an isotropic contribution from the mass of the card, and an anisotropic part due to the pressure of the air set in motion by the card. When the card moves broadside on, it experiences a much larger force of inertial resistance than when it moves end on, so that the effective force due to inertial effects is dependent on the size and shape of the card. In addition, vorticity is generated and shed in a boundary layer close to the falling card. This produces a net circulation around the card as it moves out of its own wake. These unsteady vortical flows[4,15] complicate any

analysis. In the following, we present a simple argument that captures the scaling law.

After the card reaches steady state its average terminal velocity V is constant, reflecting a balance between gravity and fluid drag. Since $w \gg d$, the card's time-averaged cross-sectional area is simply wl so that the drag force on the card scales as $\rho_f w l V^2$. Here we have implicitly made two assumptions: (1) the time scale of rotation is faster than that of translation, which is not always true, see Fig. 1; (2) the time scale associated with falling end on is much smaller than that due to either rotation or broadside-on motion. On average the drag force is balanced by the force due to gravity $(\rho_s - \rho_f) w l d g$. This yields a classical scaling law for the downward velocity $V \sim [(\rho_s/\rho_f - 1) g d]$ implied in Ref. 1 and verified in this context in Refs. 4 and 15. Composite video images such as those in Fig. 1 suggest that the tumbling rate Ω and the average velocity V are related directly according to $V \sim \Omega w$. This would correspond to a card rolling down an imaginary inclined plane, but with the opposite sense of rotation! Using this kinematic relation, the balance of forces results in the scaling law $\Omega \sim [(\rho_s/\rho_f - 1) g d]^{1/2}/w$ which matches the experimental data.

As mentioned, several effects such as the anisotropic added mass of a thin card and boundary-layer phenomena leading to flow separation and vortex shedding are responsible for the unsteady dynamics of fluid flow induced by and coupled to the motion of the card. While simple theories for the motion of a solid through a perfect fluid[19,13] explain the transition to tumbling qualitatively, they are not consistent with the scaling law for the tumbling frequency. Only further experiments involving flow visualization, measurements of fluid velocities in the wake of the tumbling card, and numerical solutions of the unsteady fluid equations can help us to better understand the deceptively simple picture of tumbling sketched in this note.

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